ELSEVIER

Journal of Geometry and Physics 16 (1995) 1–14

JOURNAL OF GEOMETRY AND PHYSICS

Classical string dynamics with non-trivial topology

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Received 26 January 1993; revised 4 November 1993

Abstract

The possibility of branching processes for classical strings is investigated on the basis of the Nambu–Goto action. We parametrize the world sheet by a Riemann surface M and introduce a degenerate, semi-Riemannian metric η on M. Well-known results about the conformal group $\mathrm{Diff}(S^1)\times\mathrm{Diff}(S^1)$ are generalized to the case of (M,η). We provide an infinite dimensional Hamiltonian setting for branching processes of strings. Finally, the classical background for the theory of quantum strings as developed by Krichever and Novikov is discussed within this classical framework.

Keywords: Degenerate semi-Riemannian manifolds; Infinite dimensional Hamiltonian systems;
1991 MSC: 53 C 50, 70 G 35, 83 E 30

Introduction

Classical string dynamics based on variational principles is a well-established theory [1]. The world sheet $\Sigma \subset \mathbb{R}^4$ of a closed string can be described by a differentiable map $x : \mathbb{R} \times S^1 \to \mathbb{R}^4$: at each moment of time $t \in \mathbb{R}$ the string is parametrized by S^1. The possibility of string branchings has been discussed in the context of cosmic strings [2] whereas this aspect has not been investigated in detail for classical fundamental strings. In the latter case the underlying idea is to treat a string which self-intersects as consisting of two strings that obey their own dynamics [3]. We study such branching processes in Section 1. From the principle of least action we derive a local criterion whether a branching solution is preferred or not.

In general, a world sheet Σ is not a differentiable manifold, but one can associate a "parameter manifold" M (a Riemann surface that generalizes $\mathbb{R} \times S^1$) to a branching process of a closed string. The world sheet Σ is then the image of M, $\Sigma = x(M)$. In Section 2 we equip M with a degenerate, semi-Riemannian metric η. The space (M,η) turns out to be the significant geometric object to study. We show that the conformal

SSDI 0393-0440(94)00018-Y

group of (M, η) is infinite dimensional, and the associated conformal algebra splits into two commuting parts.

A Hamiltonian description for branching processes is developed in Section 3. The classical phase space P is substituted by a collection of spaces $\mathcal{P} = \bigcup_{\tau\in\mathbb{R}} P_\tau$, where each P_τ is a function space over a manifold $C_\tau \subset M$, that consists of several circles. A string motion is represented by a section $s : \mathbb{R} \to \bigcup_{\tau\in\mathbb{R}} P_\tau$, i.e. $s(\tau) \in P_\tau$. For vector fields ξ on M we investigate the Poisson algebra of functions $Q_\xi(\tau) \in C^\infty(P_\tau)$. The Poisson bracket is given by $\{Q_\xi, Q_\rho\}_\tau = -Q_{[\xi_c,\rho_c]}(\tau)$ where the field ξ_c denotes the "conformal extension" of $\xi|_{C_\tau}$. For a conformal vector field ξ the function $Q_\xi(\tau)$ is a conserved quantity, and because of $\xi_c = \xi$ the conformal algebra is represented on P_τ.

In the last Section we provide a framework for a classical version of the theory of quantum strings [4,5]. The algebra of holomorphic vector fields on M is represented on P_τ, due to the fact that the conformal extension ξ_c of a holomorphic vector field yields a "local Wick rotation" on the Riemann surface M.

1. Generalized string dynamics

First, we set up notations and recall some facts about classical closed strings in Minkowski space $(\mathbb{R}^4, g = \mathrm{diag}(-1,1,1,1))$. A world sheet Σ is described by functions $x^\mu(\sigma^1, \sigma^2)$ with local parameters (σ^1, σ^2). String dynamics can be defined by the action

$$S = -\int_\Sigma \sqrt{-|h|}\, d^2\sigma\,, \quad |h| = \det(h_{\alpha\beta})\,, \tag{1.1}$$

where $h_{\alpha\beta} = g(\partial_\alpha x, \partial_\beta x) \equiv x_{,\alpha}x_{,\beta}$ are the components of the induced metric h on Σ. The Euler-Lagrange equations for x^μ are given by (we use summation convention)

$$\partial_\alpha(\sqrt{-|h|}\, h^{\alpha\beta}\partial_\beta x^\mu) = 0\,, \quad \mu = 0, \ldots, 3\,. \tag{1.2}$$

These equations are only well-defined for $|h| \neq 0$. To recover what happens at points p_d where $|h|(p_d) = 0$ holds, we consider *conformal parameters* (τ, σ). Then the induced metric h reads $h_{\alpha\beta}(\tau,\sigma) = \Lambda(\tau,\sigma)\cdot\eta_{\alpha\beta}$, with $\eta = \mathrm{diag}(-1,1)$ and some function $\Lambda \geq 0$. With $\dot{x}^\mu = \partial x^\mu/\partial\tau$ and $x'^\mu = \partial x^\mu/\partial\sigma$ the equations $h_{\alpha\beta} = \Lambda\cdot\eta_{\alpha\beta}$ are equivalent to

$$\dot{x}^2 + x'^2 = 0\,, \quad \dot{x}x' = 0\,, \tag{1.3}$$

and $\alpha = \beta = 0$ yields $\Lambda = -\dot{x}^2$. The map $\tau \to \boldsymbol{x}(\tau,\sigma) := (x^1, x^2, x^3)(\tau,\sigma)$ describes *the curve of a point* σ in $\mathbb{R}^3$, and $\boldsymbol{x}\cdot\boldsymbol{y} = x^1y^1 + x^2y^2 + x^3y^3$ denotes the Euclidean scalar product. From $|h| = -\Lambda = \dot{x}^2$ it follows that *h is singular at (τ,σ) if and only if σ moves at the velocity of light* [1]. For conformal parameters and $\Lambda \neq 0$ Eqs. (1.2) simplify to

[1] At such a point the singular behaviour of (1.2) is reflected by a singular behaviour of the world sheet: in general there is a cusp at p_d. Examples are pictured in Ref. [2].

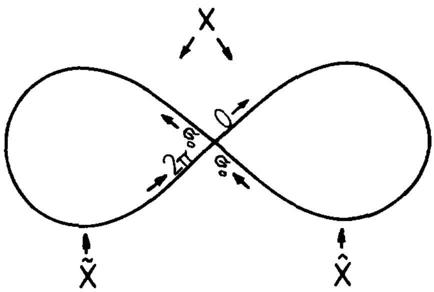

Fig. 1. Intersecting string branches.

$$\partial_\alpha \left(\frac{1}{\Lambda} \Lambda \eta^{\alpha\beta} \partial_\beta x^\mu \right) = \Box x^\mu = 0 \,, \tag{1.4}$$

with $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$. Notice that all singular terms drop out in (1.4).

For any fixed Lorentz frame a conformal (τ,σ)-parametrization is associated to a world sheet Σ as follows. To obtain τ one slices Σ with the hyperplane $x^0 = R \cdot \tau$, and the σ-parameter is fixed by (1.3) up to a shift $\sigma' = \sigma + \sigma_0$, and up to a reflection $\sigma' = -\sigma$. One can choose $R \in \mathbb{R}_+$ such that σ is 2π-periodic. The explicit representation of 2π-periodic solutions of (1.4) ($S^1 \cong \mathbb{R}/2\pi$) is useful for geometric considerations:

The unique solution of $\Box x^\mu = 0$ with respect to C^2-smooth initial data on S^1, $x^\mu(0,\sigma) = x_0^\mu(\sigma)$ and $\dot{x}^\mu(0,\sigma) = u_0^\mu(\sigma)$, is given on $\mathbb{R} \times S^1$ by

$$x^\mu(\tau,\sigma) = \tfrac{1}{2}\{x_0^\mu(\sigma+\tau) + x_0^\mu(\sigma-\tau)\} + \frac{1}{2}\int_{\sigma-\tau}^{\sigma+\tau} u_0^\mu(\sigma')\, d\sigma' \,, \qquad \mu = 0,\dots,3 \,. \tag{1.5}$$

Now suppose that two branches of the string intersect, $\boldsymbol{x}(0,\tau_0) = \boldsymbol{x}(\sigma_0,\tau_0)$, as in Fig. 1. The solution (1.5) for $x^\mu(\tau,\sigma)$ implies that both branches of the string move through each other without any influence. However, we can take a different point of view; namely that we are given two strings at τ_0, and each one obeys its own dynamics for $\tau \geq \tau_0$:

$$\begin{aligned}
\hat{x}^\mu(\tau,\sigma) &= \tfrac{1}{2}\{y^\mu(\sigma+\tau-\tau_0) + y^\mu(\sigma-\tau+\tau_0)\} \\
&\quad + \frac{1}{2}\int_{\sigma-\tau+\tau_0}^{\sigma+\tau-\tau_0} r^\mu(\sigma')d\sigma' \,, \quad \sigma \in [0,\sigma_0) \,, \\
\tilde{x}^\mu(\tau,\sigma) &= \tfrac{1}{2}\{z^\mu(\sigma+\tau-\tau_0) + z^\mu(\sigma-\tau+\tau_0)\} \\
&\quad + \frac{1}{2}\int_{\sigma-\tau+\tau_0}^{\sigma+\tau-\tau_0} s^\mu(\sigma')d\sigma' \,, \quad \sigma \in [\sigma_0,2\pi) \,.
\end{aligned} \tag{1.6}$$

The functions $y^\mu(\sigma) = x^\mu(\tau_0,\sigma)$ and $r^\mu(\sigma) = \dot{x}^\mu(\tau_0,\sigma)$ have periodicity σ_0, and analogously z^μ and s^μ are $(2\pi - \sigma_0)$-periodic functions. The equations $\Box\hat{x}^\mu = \Box\tilde{x}^\mu = 0$ are valid only in a distributional sense because of the kinks which arise from the intersection point. We remark that no conservation law (energy, momentum, ...) is violated by the splitting. When the points σ_0 and 0 move at the velocity of light Eqs. (1.2) are *not* well-defined because of $h(p_d) = 0$. Outside this point both solutions (1.5) and (1.6) are well-defined, hence there is no natural preference for one of them.

We thus consider how the action behaves as a function of τ ($\tau_0 := 0$, for convenience). The on-shell action $S_o = \int \dot{x}^2 d^2\sigma$ follows from (1.1) and (1.3). We examine the difference of

$$S_{\text{free}}(\tau) := \int_0^\tau \int_0^{2\pi} \dot{x}^2 d\sigma d\tau'$$

and

$$S_{\text{split}}(\tau) := \int_0^\tau \int_0^{\sigma_0} \dot{\hat{x}}^2 d\sigma d\tau' + \int_0^\tau \int_{\sigma_0}^{2\pi} \dot{\tilde{x}}^2 d\sigma d\tau' :$$

Proposition 1.1. *$\Delta S(\tau) := S_{\text{free}}(\tau) - S_{\text{split}}(\tau)$ has derivatives $(d\Delta S/d\tau)(0) = 0$ and*

$$\frac{d^2\Delta S}{d\tau^2}(0) = \big(\boldsymbol{u}_0(\sigma_0) - \boldsymbol{u}_0(0)\big)^2 - \big(\boldsymbol{x}_0'(\sigma_0) - \boldsymbol{x}_0'(0)\big)^2 . \tag{1.7}$$

Proof. The next formula follows from the fact that the values of $x^\mu(\tau,\sigma)$ for the solutions (1.5) and (1.6) coincide outside the σ-intervals $[-\tau,\tau]$ and $[\sigma_0-\tau,\sigma_0+\tau]$:

$$\begin{aligned}\frac{d\Delta S}{d\tau}(\tau) = &\int_{-\tau}^{\tau} [\dot{x}^2(\tau,\sigma) + \dot{x}^2(\tau,\sigma_0+\sigma)]d\sigma \\ &- \int_0^\tau [\dot{\hat{x}}^2(\tau,\sigma) + \dot{\hat{x}}^2(\tau,\sigma_0-\tau+\sigma)]d\sigma \\ &- \int_{-\tau}^0 [\dot{\tilde{x}}^2(\tau,\sigma) + \dot{\tilde{x}}^2(\tau,\sigma_0+\tau+\sigma)]d\sigma .\end{aligned}$$

Thus, $\Delta\dot{S}(0) = 0$. One has to be cautious with terms like $(d/d\tau)\int_0^\tau \dot{\hat{x}}^2(\tau,\sigma)d\sigma$ since $\dot{\hat{x}}^2(\tau,\sigma)$ is discontinuous at $\sigma = \tau$. We find

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \int_0^\tau \dot{\hat{x}}^2(\tau,\sigma)d\sigma = \lim_{\epsilon\downarrow 0} \dot{\hat{x}}^2(\epsilon,0) .$$

Differentiating $\Delta\dot{S}(\tau)$ and collecting terms from (1.5), (1.6) yields $\Delta\ddot{S}(0) = (u_0(\sigma_0) - u_0(0))^2 - (x_0'(\sigma_0) - x_0'(0))^2$. Since the time component of this expression vanishes we obtain (1.7). □

A possible modification of string dynamics is provided by the requirement of *least growth of* $S(\tau)$. This will then imply branching effects of strings on a classical level. For example, for $\Delta\ddot{S}(0) > 0$ the splitting solution is preferred. Thus the first term in (1.7) has the following effect: at high relative velocities $|\boldsymbol{u}(\sigma_0) - \boldsymbol{u}(0)|$ a splitting occurs, whereas at low velocities there is no "interaction". To see what the second term in (1.7) means, suppose that the two string branches are locally straight lines, and $\boldsymbol{u}_0 \equiv 0$ holds. In the case of solution (1.6) two kinks on each string move away from the intersection point. The points σ between these kinks move at the speed $\frac{1}{2}|\boldsymbol{x}_0'(\sigma_0) - \boldsymbol{x}_0'(0)|$, thus $\Delta S(\tau)$ decreases.

Remarks. 1. This kind of dynamics is not invariant under time reversal: Suppose a splitting occurs at $\tau = 0$. Then the time-reversed motion is a process where two strings merge into one. However, the minimality criterion also enforces a splitting for $\tau < 0$, since at $\tau = 0$ only $\boldsymbol{u}_0$ changes to $-\boldsymbol{u}_0$, which has no effect on (1.7).

2. One can show that the sign of $\Delta\ddot{S}(0)$ does not depend on the Lorentz frame in Minkowski space, if such frames have the same time orientation. In that sense the dynamical criterion is Lorentz invariant.

So far we discussed solutions of the form (1.6) but it is obvious how to construct merging solutions of the same type. We only have to perform some shifts $\sigma_i \to \sigma_i + \Theta_i$ along each string in order to get a continuous parametrization of that merging solution.

2. The geometry of parameter manifolds

First, we describe how branching processes can be parametrized by Riemann surfaces. Consider the history of n classical strings which do not interact for $\tau < \tau_1$. At times $\tau_1, \ldots, \tau_k$ string branchings occur, and the configuration for $\tau > \tau_k$ consists of m strings. It has been shown [6] that "light cone diagrams" can be parametrized by Riemann surfaces M and the construction of M given there applies to our picture of string motion. In addition, the abstract evolution parameter τ in Ref. [6] is now related to physical time by $t = R \cdot \tau$. The "position" of the strings on M at fixed time τ_0 is described by

$$C_{\tau_0} = \{q \in M \mid \tau(q) = \tau_0\}, \quad \tau(q) = \mathrm{Re}\left(\int_{q_0}^{q} \kappa\right), \tag{2.1}$$

where q_0 is a fixed reference point and κ is a holomorphic differential on M. The *critical points* p_c of the function $\tau : M \to \mathbb{R}$, Eq. (2.1), are the zeros of κ, and the set $\{p_c\}$

describes all points where the strings split or rejoin, i.e., $\tau_c := \tau(p_c) \in \{\tau_1, \ldots, \tau_k\}$. The σ-parameter is given by $\sigma = \mathrm{Im} \int_{q_0}^{q} \kappa$ modulo some angle σ_i along each string and modulo some twist angle if two strings merge, due to the fact that κ has purely imaginary periods. Thus, if $q_0 \in M$ is not a critical point, the parameters $u = \tau + i\sigma = \int_{q_0}^{q} \kappa$ define a local holomorphic chart around q_0. An essential observation is that these charts consist of conformal parameters for the string world sheet. We call (M, κ) the *parameter manifold* of Σ.

Next, we introduce a degenerate metric on M. This is motivated by the following observation. In a conformal chart $V \subset \mathring{M} := M \backslash \{p_c\}$ the action (1.1) can be written as

$$\tilde{S}_V = -\frac{1}{2} \int_V \eta^{\alpha\beta} \partial_\alpha x \partial_\beta x \, d^2\sigma , \tag{2.2}$$

and the equations of motion (1.4) follow from this action. $\tilde{S}_V$ contains the flat metric η and is locally (i.e. for any domain $V \subset \mathring{M}$) invariant under *conformal transformations*. These are diffeomorphisms $\Psi : V \to V' \subset \mathring{M}$, satisfying $\Psi^*\eta = F \cdot \eta$ with a smooth function F. This symmetry property of $\tilde{S}_V$ implies conservation laws for the equations of motion. In that sense the metric η is of physical importance and not the pullback metric $-\dot{x}^2 \cdot \eta$.

Lemma 2.1. *The metric $\eta = -d\tau^2 + d\sigma^2$ on $\mathring{M}$ has a unique C^∞-smooth extension to M. It is given by $\eta(p_c) = 0$, for all critical points p_c on M.*

Proof. We represent η in a chart around p_c with coordinates (τ', σ'), which obey $\tau + i\sigma - a = (\tau' + i\sigma')^n$, with $n \geq 2$, $a \in \mathbb{C}$ and $p_c \cong (0,0)$ [6]. Then $\partial\sigma^\alpha / \partial\sigma^{\alpha'}$ depends analytically on τ', σ' and vanishes in the limit $p \to p_c$. The assertion now follows from $\eta_{\alpha'\beta'} = (\partial\sigma^\alpha/\partial\sigma^{\alpha'})(\partial\sigma^\beta/\partial\sigma^{\beta'})\eta_{\alpha\beta}$. □

Definition. The *conformal group of* (M, η) consists of the diffeomorphisms $\Psi : M \to M$ which obey $\Psi^*\eta = F \cdot \eta$, where $F \neq 0$ denotes a smooth function on M.

We now consider a one-parameter group of conformal diffeomorphisms Ψ_s, i.e. a *conformal flow* $\{\Psi_s | s \in \mathbb{R}\}$. Because of $\Psi_0 = \mathrm{id}_M$ and $\Psi_s^*\eta = F_s \cdot \eta$, the critical points p_c are fixed points of Ψ_s. The *conformal vector field* $\xi = (d\Psi_s/ds)|_{s=0} = \dot{\Psi}_0$ generates this flows on M, and $\xi(p_c) = \dot{\Psi}_0(p_c) = 0$ holds. The Lie algebra of conformal vector fields on (M, η) is denoted by C_M. It is well-known that in conformal parameters $\xi = \xi^0\partial_\tau + \xi^1\partial_\sigma \in C_M$ obeys

$$\xi^0_{,0} = \xi^1_{,1} , \quad \xi^1_{,0} = \xi^0_{,1} . \tag{2.3}$$

From $\Psi_s^*\eta = F_s \cdot \eta$ we do not get any conditions on the derivatives of ξ at p_c because the resulting equations are trivial (0=0) due to the vanishing of η at p_c.

To examine the conformal group we will need the set Γ of lightlike geodesics γ on $\mathring{M}$. It is a simple fact that Γ consists of two parts Γ^+ and Γ^-: geodesics $\gamma^\pm \in \Gamma^\pm$ satisfy $\sigma^\pm = \text{const.}$ in local light cone coordinates $\sigma^\pm = \tau \pm \sigma$ around each $p \in \mathring{M}$.

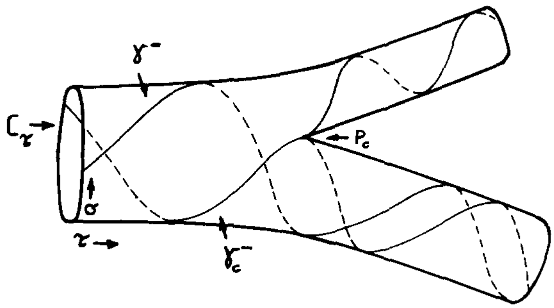

Fig. 2. Light curves covering M.

The curves $\gamma^\pm$ arising from C_τ cover $\mathring{M}$ since along a fixed "string tube" $[\tau_a, \tau_b] \times S^1$ these curves wind up with a constant slope of $\Delta\tau/\Delta\sigma = \mp 1$ (an example is pictured in Fig. 2). The exceptional curves $\gamma_c^\pm$ converge into critical points p_c or emerge from them.

The general solution of (2.3) in $\sigma^\pm$-coordinates reads $\xi = f(\sigma^+)\partial_{\sigma^+} + g(\sigma^-)\partial_{\sigma^-}$, with $\partial_{\sigma^\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. Of course, this local decomposition of ξ into a ∂_{σ^+} and a ∂_{σ^-}-part defines a unique global decomposition $\xi = \xi^+ + \xi^-$ with Lie bracket $[\xi^+, \xi^-] = 0$.

Theorem 2.2. *C_M consists of two commuting, infinite dimensional subalgebras: $C_M = C_M^+ \oplus C_M^-$. Each $\xi = \xi^+ + \xi^- \in C_M$ is uniquely determined by an arbitrary restriction $\xi|_{C_\tau}$ (2.1), and ξ can be obtained by parallel transport of $\xi^\pm|_{C_\tau}$ along the light curves $\gamma^\pm$.*

Proof. In a neighbourhood of any point $p \neq p_c$ we have $\xi = f(\sigma^+)\partial_{\sigma^+} + g(\sigma^-)\partial_{\sigma^-}$, and the coefficients $f(\sigma^+)$ and $g(\sigma^-)$ are constant along the curves γ^+ and γ^-, respectively. Thus ξ^+ and ξ^- coincide with those fields which arise from $\xi^\pm|_{C_\tau}$ by parallel transport along $\gamma^\pm$ (with respect to the metric η). On a curve γ_c^+ the field ξ^+ must vanish; otherwise $\xi(p_c)$ would be singular since $\partial\sigma^{\alpha'}/\partial\sigma^\alpha$ diverges at p_c, cf. Lemma 2.1. An analogous argument holds for γ_c^-, ξ^-. Therefore, the components $\xi^\pm$ are fixed by this construction on the whole of M. The zeros of $\xi^\pm$ along $\gamma_c^\pm$ impose finitely many vanishing conditions on $\xi^\pm|_{C_\tau}$, say at the points $p_1, \ldots, p_l \in C_\tau$. From a smooth function f on C_τ with compact support on $C_\tau \setminus \{p_1, \ldots, p_l\}$ we get the vectors $f(\sigma)\partial_{\sigma^+}|_{C_\tau}$. Parallel transport of these vectors along γ^+ yields a global conformal vector field ξ^+ on M. Since the vector space of such functions f on C_τ is infinite dimensional the same holds for C_M^+, and also for C_M^-. □

An arbitrary vector field ξ_0 along C_{τ_0} ($\tau_0 \neq \tau_c$) splits into $\xi_0 = \xi_0^+\partial_{\sigma^+} + \xi_0^-\partial_{\sigma^-}$. By parallel transport of $\xi_0^\pm$ along $\gamma^\pm$ one obtains the field $\xi_c = \xi^+ + \xi^-$ on M. In general, ξ_c is discontinuous at the points $\gamma_c^\pm$. To be definite we set $\xi_c = 0$ along $\gamma_c^\pm$.

Definition. The field ξ_c, obtained from ξ_0, is called the *conformal extension* of ξ_0.

Theorem 2.3. *The conformal group of (M, η) is infinite dimensional.*

Proof. It suffices to show that for each generator $\xi \in C_M$ the conformal flow $\Psi_s = \exp(s\xi)$ exists on M for all $s \in \mathbb{R}$. Recall the set $\{\tau_1, ..., \tau_k\}$ where branchings occur. Each time slice C_τ with $\tau \in (\tau_k, \infty)$ consists of m disjoint components C_τ^j, $j = 1, \ldots, m$. The subspaces $M_j := \bigcup_{\tau \in (\tau_k, \infty)} C_\tau^j \subset M$ are parametrized by holomorphic coordinates $u = \tau + i\sigma$ with $\sigma = \sigma \bmod \sigma_j$. From $u' = v^1 + iv^2 = \exp(-2\pi u/\sigma_j)$ we obtain another holomorphic coordinate on M_j and $u' = 0$ corresponds to a point Q_j outside M. We also obtain additional points P_i from $N_i = \bigcup_{\tau \in (-\infty, \tau_1)} C_\tau^i$, $i = 1, \ldots, n$. In this way a holomorphic compactification $\overline{M} = M \cup \{Q_1, \ldots, Q_m, P_1, \ldots, P_n\}$ is defined. One verifies that the components of $\xi = \xi^1 \partial_{v^1} + \xi^2 \partial_{v^2}$ vanish in the limit $(v^1, v^2) \to 0$, since $\xi^\pm$ is bounded on $\mathring{M}$. We can thus extend continuously ξ to $\overline{\xi}$ by the definition $\overline{\xi}(Q_j) = 0$, $j = 1, \ldots, m$. Analogously we set $\overline{\xi}(P_i) = 0$. The continuous field $\overline{\xi}$ is defined on the compact Riemann surface $\overline{M}$, so it generates a global flow $\{\overline{\Psi}_s | s \in \mathbb{R}\}$ on $\overline{M}$. The conformal flow on M is now given by $\exp(s\xi) = \overline{\Psi}_s|_M$. □

In the next section we need the geometric structure of the sets C_τ. If $\tau \notin \{\tau_1, \ldots, \tau_k\}$ then C_τ is a smooth submanifold of M. C_τ is naturally isomorphic to $C_{\tau'}$ for τ, $\tau' \in (\tau_i, \tau_{i+1})$, $i = 0, \ldots, k$ $(\tau_0 = -\infty, \tau_{k+1} = \infty)$. The isomorphism $I_{\tau\tau'} : C_\tau \to C_{\tau'}$ is obtained by an identification of points $p \in C_\tau$ with $p' \in C_{\tau'}$, lying on the same integral curve of the vector field ∂_τ on $\mathring{M}$. A *critical curve* C_{τ_i} is not a smooth manifold but $\mathring{C}_{\tau_i} := C_{\tau_i} \backslash \{p_c\}$ can be embedded into C_τ for $\tau \in (\tau_{i-1}, \tau_i)$, and also for $\tau \in (\tau_i, \tau_{i+1})$, by the same identification as above. Therefore, two natural compactifications of $\mathring{C}_{\tau_i}$ exist: the compactified space $C_{\tau_i >}$ is isomorphic to C_τ, $\tau \in (\tau_{i-1}, \tau_i)$, and $C_{\tau_i <}$ is isomorphic to C_τ, $\tau \in (\tau_i, \tau_{i+1})$. For example, in Fig. 2 one obtains one circle S^1 for $C_{\tau_1 >}$ and two copies of S^1 for $C_{\tau_1 <}$. By $I_{\tau\tau_i} : C_\tau \to C_{\tau_i >}$, $\tau \in (\tau_{i-1}, \tau_i)$, we denote the first identifying isomorphism.

3. A Hamiltonian description for branching processes

Before we consider the theory with varying string topology we set up the general context needed. A Hamiltonian description for the classical string starts with

$$H(x, \pi) = \frac{1}{2} \int_{S^1} (x'^2 + \pi^2) d\sigma \tag{3.1}$$

together with the constraints $\pi x' = 0$, $\pi^2 + x'^2 = 0$. The domain of H is the Hilbert space $P := (\bigoplus_{\mu=0}^3 H^1(S^1)) \oplus (\bigoplus_{\mu=0}^3 L^2(S^1))$, which is an orthogonal sum of Hilbert spaces. $H^1 \subset L^2$ is the first (real) Sobolev space and P is denoted by $H^1(S^1)^4 \times L^2(S^1)^4$, for short. The canonical weak symplectic form $\omega : TP \times TP \to \mathbb{R}$ is defined by (cf. Ref. [7])

$$\omega_{(x,\pi)}\big((V, W); (Y, Z)\big) := \langle V^\mu, Z_\mu \rangle_{L^2} - \langle Y^\mu, W_\mu \rangle_{L^2} . \tag{3.2}$$

Since ω is a weak form, the defining equation for Hamiltonian vector fields X_f,

$$df(V,W) = \omega(X_f;(V,W)) \quad \forall (V,W) \in TP\,, \tag{3.3}$$

need not have a solution at each $(x,\pi) \in P$. Poisson brackets of C^1-functions f, g on P, $\{f,g\} := \omega(X_f;X_g)$, are defined on the common domain $D = D_f \cap D_g$ of X_f and X_g. The domain of $X_H = (\pi, x'')$ is given by $D_H = H^2(S^1)^4 \times H^1(S^1)^4$, which is a dense subspace of P. The solution $(x,\pi)_\tau$ of the Hamiltonian equation of motion, $(\dot{x},\dot{\pi}) = X_H$, starting at $(x,\pi)_0$ is given by (1.5) together with $\pi = \dot{x}$. For each τ the linear map $\tilde{\Phi}_\tau : (x,\pi)_0 \to (x,\pi)_\tau$ is continuous on $H^2(S^1)^4 \times H^1(S^1)^4$. Its unique continuous extension yields the Hamiltonian flow of X_H on P, $\Phi_\tau : P \to P$.

To describe branching processes it is natural to substitute S^1 by C_τ in (3.1). The generalization of P then reads $P_\tau := H^1(C_\tau)^4 \times L^2(C_\tau)^4$, where the integration measure is defined by the conformal parameter σ. Since the critical curves C_{τ_i} are not smooth manifolds we set $P_{\tau_i} := H^1(C_{\tau_i>})^4 \times L^2(C_{\tau_i>})^4$. This is a natural choice since the time evolution (as explained below) of any state $(x,\pi) \in P_\tau$, $\tau \in (\tau_{i-1},\tau_i)$, yields an element of $H^1(C_{\tau_i>})^4 \times L^2(C_{\tau_i>})^4$. Different string motions are parametrized by different systems of curves C_τ. We aim to describe all possible motions which can be parametrized by a prescribed, fixed system[2]. For that purpose we will restrict each P_τ to a subset P_τ^r such that the evolution of any $(x,\pi) \in P_\tau^r$ is described by these C_τ. This kind of restriction has to be added to the restriction by the constraints $x'^2 + \pi^2 = x'\pi = 0$.

Definition. The *symplectic form ω_τ on P_τ* is given by formula (3.2) where $\langle\ ,\ \rangle_{L^2}$ denotes the scalar product in $L^2(C_\tau)$, resp. in $L^2(C_{\tau_i>})$. The *Poisson bracket* reads $\{f,g\}_\tau = \omega_\tau(X_f,X_g)$ with $X_f, X_g \in TP_\tau$. We call $\mathcal{P} = \bigcup_{\tau\in\mathbb{R}}(P_\tau,\omega_\tau)$ *the generalized phase space.*

Recall the isomorphisms $I_{\tau\tau'}$, which are defined for $\tau,\tau' \in J_0 = (-\infty,\tau_1]$, $\tau,\tau' \in J_i = (\tau_i,\tau_{i+1}]$ with $i = 1,\dots,k-1$, or $\tau,\tau' \in J_k = (\tau_k,\infty)$. By pullback they induce Hilbert space isomorphisms $I^*_{\tau\tau'} : P_{\tau'} \to P_\tau$. A string motion for $\tau \in J_i$ with initial state $(x,\pi)_{\tau_0} \in P_{\tau_0}$ is described as follows. The vector field $(\pi,x'')_{\tau_0}$ defines the Hamiltonian flow Φ^i_u on P_{τ_0}. We identify $\Phi^i_u(x,\pi)_{\tau_0}$ with $I^*_{\tau_0+u,\tau_0}(\Phi^i_u(x,\pi)_{\tau_0}) \in P_{\tau_0+u}$, as long as $\tau_0 + u \in J_i$ holds. In this way a section $s_i : J_i \to \bigcup_{\tau\in J_i} P_\tau$ with a prescribed value at τ_0 is obtained.

We now define the "transition" of a function f on $C_{\tau_i>}$ to a function $\tilde{f}$ on $C_{\tau_i<}$: we restrict f to $\mathring{C}_{\tau_i} \subset C_{\tau_i>}$, and then extend this restricted function arbitrarily to $C_{\tau_i<} \supset \mathring{C}_{\tau_i}$. In this way a function $\tilde{f}$ is defined modulo changes on a set of Lebesgue measure zero. We obtain by this procedure a Hilbert space isomorphism $F_i : L^2(C_{\tau_i>})^4 \times L^2(C_{\tau_i>})^4 \to L^2(C_{\tau_i<})^4 \times L^2(C_{\tau_i<})^4$. Notice that F_i describes the physical re-interpretation of the string configuration.

[2] Such systems $\{C_\tau\}$ can be defined on any compact Riemann surface $\overline{M}$ by (2.1). The existence of κ with prescribed properties is well-known, cf. Ref. [8].

Lemma 3.1. *The image of* $(x,\pi) \in H^1(C_{\tau_i>})^4 \times L^2(C_{\tau_i>})^4$ *under* F_i *is an element of* $H^1(C_{\tau_i<})^4 \times L^2(C_{\tau_i<})^4$ *if and only if the x-component of* $F_i(x,\pi)$ *is continuous at* $C_{\tau_i<}\backslash\mathring{C}_{\tau_i}$.

Proof. We can identify an H^1-function on a one-dimensional space with its unique continuous representative, due to the Sobolev Lemma. $(x,\pi) \in H^1(C_{\tau_i>})^4 \times L^2(C_{\tau_i>})^4$ implies continuity of the x-component of $F_i(x,\pi)$ on $\mathring{C}_{\tau_0}$. The remaining points to consider are $C_{\tau_i<}\backslash\mathring{C}_{\tau_i}$. The assertion now follows from the Sobolev Lemma. □

For $\tau_0 < \tau_1$ the evolution of $(x,\pi)_{\tau_0} \in P_{\tau_0}$ is described by $s_0 : (-\infty,\tau_1] \to \bigcup_{\tau\in(-\infty,\tau_1]} P_\tau$. We want to parametrize the time evolution by the curves C_τ. Since $C_{\tau_1<}$ is isomorphic to C_τ, $\tau \in (\tau_1,\tau_2]$, we must somehow interpret $s_0(\tau_1)$ as a function on $C_{\tau_1<}$. From the physical picture it is natural to do this by the isomorphism F_1. However, we obtain the *transition condition* $s_0(\tau_1) \in F_1^{-1}(H^1(C_{\tau_1<})^4 \times L^2(C_{\tau_1<})^4)$, because only in this case there is a well-defined time evolution of $F_1(s_0(\tau_1))$, for $\tau \in [\tau_1,\tau_2]$. This is physically reasonable due to Lemma 3.1: only if the x-component is continuous along $C_{\tau_1<}$ ($\simeq$ closed strings) one obtains a well-defined motion. If $s_0(\tau_1)$ does not satisfy the transition condition we exclude the initial state $(x,\pi)_{\tau_0}$ from P_{τ_0}. Inductively, we obtain further restrictions on P_{τ_0} by the conditions $s_{i-1}(\tau_i) \in F_i^{-1}(H^1(C_{\tau_i<})^4 \times L^2(C_{\tau_i<})^4)$. Any state of the resulting set $P^r_{\tau_0} \subset P_{\tau_0}$ has a time evolution that is described by the curves C_τ. For arbitrary $\tau \in \mathbb{R}$ the set P^r_τ is now obtained by the dynamical evolution of $P^r_{\tau_0}$.

Summarizing, a string motion on (M,κ) is given by a section $s : \mathbb{R} \to \bigcup_{\tau\in\mathbb{R}} P_\tau$, and the set of states at fixed τ has to be restricted as described. Notice that by "section" we only mean $s(\tau) \in P_\tau$. The set $\bigcup_{\tau\in\mathbb{R}} P_\tau$ has no canonical smooth vector bundle structure because the spaces P_τ and $P_{\tau'}$ are not naturally isomorphic for arbitrary τ,τ'.

Next, we examine the Poisson algebra of functions that are related to conformal transformations. Due to Noether's theorem for each conformal vector field ξ the conserved quantity $Q_\xi(\tau) = \int_{C_\tau} \xi^\alpha T_\alpha{}^0 d\sigma$ is associated, i.e. $Q_\xi(\tau) = \text{const.}$ for any solution of (1.4). The canonical energy momentum tensor $T_\alpha{}^\beta = x_{,\alpha}x^{,\beta} - \frac{1}{2}\delta^\beta_\alpha x_{,\gamma}x^{,\gamma}$ is derived from $\tilde{S}$, Eq. (2.2). In phase space variables the function $Q_\xi(\tau) : P_\tau \to \mathbb{R}$ reads

$$Q_\xi(\tau)(x,\pi) = -\int_{C_\tau} [\tfrac{1}{2}\xi^0(\pi^2 + x'^2) + \xi^1\pi x']d\sigma. \tag{3.4}$$

Theorem 3.2. *An anti-representation of the algebra* $\mathcal{C}_M$ *is defined on* $H^2(C_\tau)^4 \times H^1(C_\tau)^4$ *by Poisson brackets of the functions (3.4):*

$$\{Q_\xi, Q_\rho\}_\tau = -Q_{[\xi,\rho]}(\tau), \quad \forall \xi,\rho \in \mathcal{C}_M, \ \forall \tau \in \mathbb{R}. \tag{3.5}$$

Proof. The Fréchet derivative of $Q_\xi(\tau) \in C^\infty(P_\tau)$ is given by

$$dQ_\xi(x,\pi)(V,W) = -\int_{C_\tau} \left[(\xi^0 x' + \xi^1 \pi)V' + (\xi^0 \pi + \xi^1 x')W\right] d\sigma\,,$$
$$(V,W) \in T_{(x,\pi)}P_\tau\,.$$

We integrate the first term by parts and obtain from (3.3) (with $X_{Q_\xi} = (Y_\xi, Z_\xi)$):

$$\int_{C_\tau} [(\xi^0 x' + \xi^1\pi)' \cdot V - (\xi^0\pi + \xi^1 x') \cdot W]\, d\sigma = \langle Y_\xi, W\rangle_{L^2} - \langle Z_\xi, V\rangle_{L^2}\,.$$

It follows that $(Y_\xi, Z_\xi) = -(\xi^0\pi + \xi^1 x', (\xi^0 X' + \xi^1\pi)')$, so the Poisson bracket reads

$$\begin{aligned}\{Q_\xi, Q_\rho\}_\tau &= \langle Y_\xi, Z_\rho\rangle_{L^2} - \langle Z_\xi, Y_\rho\rangle_{L^2}\\ &= \int_{C_\tau} (\xi^0\pi + \xi^1 x')(\rho^0 x' + \rho^1\pi)' d\sigma - (\xi \leftrightarrow \rho)\,,\end{aligned}$$

where $\xi \leftrightarrow \rho$ denotes the same integral with ξ and ρ interchanged. We find for $\{Q_\xi, Q_\rho\}_\tau$:

$$\begin{aligned}&\int_{C_\tau} \{(\xi^0\partial_1\rho^0 + \xi^1\partial_1\rho^1)\pi x' - \tfrac{1}{2}(\rho^0\partial_1\xi^1 - \xi^1\partial_1\rho^0)x'^2\\ &\quad + \tfrac{1}{2}(\xi^0\partial_1\rho^1 - \rho^1\partial_1\xi^0)\pi^2\} d\sigma - (\xi \leftrightarrow \rho).\end{aligned}$$

We use (2.3) to substitute the first term in each bracket, e.g. $\xi^0\partial_1\rho^0 = \xi^0\partial_0\rho^1$, and after collecting analogous expressions from $\xi \leftrightarrow \rho$ the assertion follows. □

A crucial step in the proof is the substitution of terms like $\xi^0\partial_1\rho^0$ by $\xi^0\partial_0\rho^1$ because only in this way one obtains the τ-derivatives which are present in the Lie bracket $[\xi, \rho]$. For general vector fields ξ, ρ on M this substitution is not allowed but we observe that the field $\xi|_{C_{\tau_0}}$ coincides with its conformal extension $\xi_c|_{C_{\tau_0}}$. Thus $Q_\xi(\tau_0) = Q_{\xi_c}(\tau_0)$.

Corollary 3.3. *Let ξ and ρ be arbitrary smooth vector fields on M. For $\tau \neq \tau_c$ the Poisson bracket $\{Q_\xi, Q_\rho\}_\tau$ is given on $H^2(C_\tau)^4 \times H^1(C_\tau)^4$ by*

$$\{Q_\xi, Q_\rho\}_\tau = -Q_{[\xi_c, \rho_c]}(\tau)\,, \tag{3.6}$$

where ξ_c and ρ_c denote the conformal extension of ξ, arising from C_τ. □

For $\tau = \tau_c$ and $\xi(p_c) \neq 0$ the components $\xi^\pm(p)$ diverge in the limit $p \to p_c$. We thus excluded $\tau = \tau_c$. Notice that this corollary remains valid for $\xi_p, \rho_p \in T_pM \otimes \mathbb{C}$.

4. The classical background of KN-theory

The purpose of this section is to show that holomorphic objects on M are useful tools in the Hamiltonian setting of string dynamics in Minkowski space. We consider the case of one string for $\tau < \tau_1$, and one for $\tau > \tau_k$, because (4.1) holds in this case (for generalizations, cf. Ref. [9]). We denote the space of complex valued C^∞-vector fields on C_τ by $\mathcal{L}(C_\tau)$ and the space of meromorphic vector fields on $\overline{M}$ that are holomorphic on M by $\mathcal{L}(M)$. In complex (τ,σ)-coordinates any $e \in \mathcal{L}(M)$ reads as $l(\tau+i\sigma)\frac{1}{2}(\partial_\tau - i\partial_\sigma)$. If we fix $\tau \neq \tau_c$ and use σ as coordinate along C_τ, the "restriction"

$$R_\tau : l(\tau+i\sigma)\tfrac{1}{2}(\partial_\tau - i\partial_\sigma) \to -il(\tau+i\sigma)\partial_\sigma$$

is a Lie algebra homomorphism from $\mathcal{L}(M)$ to $\mathcal{L}(C_\tau)$. A countable basis $\{e_n\}$ of $\mathcal{L}(M)$ exists [4]. Meromorphic quadratic differentials Ω on $\overline{M}$ which are holomorphic on M are represented locally by $\Omega = f(z)dz^2$. Let $\{\Omega^m\}$ be the dual basis to $\{e_n\}$ with respect to the pairing $\langle\Omega, e\rangle := \int_{C_\tau}\Omega(e,\cdot)$, i.e., $\langle\Omega^m, e_n\rangle = \delta_n^m$ [5]. For $\tau \neq \tau_c$ a smooth quadratic differential Ω_τ on C_τ has an expansion (i_τ^* is the pullback of the embedding $i_\tau : C_\tau \hookrightarrow M$)

$$\Omega_\tau = \sum_{-\infty}^{\infty} b_n \cdot i_\tau^*\Omega^n \ . \tag{4.1}$$

This expansion allows us to describe the energy momentum tensor T conveniently. In $\sigma^\pm$-coordinates T reads $T = T_{++}d\sigma^+ \otimes d\sigma^+ + T_{--}d\sigma^- \otimes d\sigma^-$, with $T_{nn} = (\partial x/\partial\sigma^n)^2$. We obtain $i_\tau^*T = T_{++}d\sigma^2 + T_{--}d\sigma^2 =: T_+ + T_-$, and represent T_+ as in (4.1), $T_+ = \sum_{-\infty}^{\infty} b_n(\tau)\cdot i_\tau^*\Omega^n$.

Proposition 4.1. *For $\tau \neq \tau_c$ the components of the T_+-part of the energy momentum tensor are given by $b_n(\tau) = \int_{C_\tau} T_+(R_\tau e_n, \cdot) =: L_{e_n}(\tau)$. They satisfy the Poisson algebra*

$$\{L_{e_n}, L_{e_m}\}_\tau = L_{[e_n,e_m]}(\tau) \ . \tag{4.2}$$

Proof. From $i_\tau^*(d\tau + id\sigma)^2 = -d\sigma^2$ we obtain for $\int_{C_\tau} T_+(R_\tau e_n,\cdot) = -\int_{C_\tau} il_n(\tau + i\sigma)T_{++}d\sigma$:

$$-\sum_m b_m \int_{C_\tau} (i_\tau^*\Omega^m)(\sigma) l_n(\tau+i\sigma) i d\sigma = \sum_m b_m \int_{C_\tau} \Omega^m(\tau+i\sigma) l_n(\tau+i\sigma) i d\sigma = b_n \ .$$

For $\xi^+ \in C_M^+$ Eq. (3.4) yields $Q_{\xi^+}(\tau) = -\int_{C_\tau}\xi^+(\tau,\sigma)T_{++}d\sigma$. Comparing Q_{ξ^+} with $L_{e_n} = -\int_{C_\tau} il_n(\tau+i\sigma)T_{++}d\sigma$ leads to the vector field $e_n^+ := il_n(\tau_0+i\sigma)\partial_{\sigma^+}$ on C_{τ_0}. Its local conformal extension in a neighbourhood of C_{τ_0} is given by

$$e^+_{n\,c}(\tau+i\sigma) = il_n(\tau_0 + i(\sigma+\Delta\tau))\partial_{\sigma^+} \ , \quad \tau = \tau_0 + \Delta\tau \ . \tag{4.3}$$

At $\tau = \tau_0$ we have $L_{e_n} = Q_{e^+_{n\,c}}$. Using (3.6) and (4.3) one calculates

$$\{L_{e_n}, L_{e_m}\}_{\tau_0} = \{Q_{e_n^+}, Q_{e_m^+}\}_{\tau_0} = -Q_{[e^+_{nc}, e^+_{mc}]} = -Q_{-[e_n,e_m]^+_c} = L_{[e_n,e_m]} \ . \qquad \square$$

The conformal extension (4.3) is obtained by a substitution in the argument of l_n, namely $(\tau_0 + \Delta\tau + i\sigma) \mapsto (\tau_0 + i(\sigma + \Delta\tau))$. In essence, this is a local Wick rotation because we only substitute $\Delta\tau$ by $i\Delta\tau$. The derivative ∂_{σ^+} acts on the conformal extension $l_n(\tau_0 + i(\sigma + \Delta\tau))$ in the same way as $i\, d/du$ acts on $l_n(u)$. In this way the complex Lie algebra structure shows up. Notice that the concept of a Wick rotation only makes sense because a *canonical splitting* of the Riemann surface into *time*$\times$*space* is induced by the differential κ.

Remark. The fields e_n obey a Lie algebra $[e_n, e_m] = \sum_j C^j_{nm} e_{n+m+j}$ with structure constants C^j_{nm}, and the summation ranges over some fixed, finite set [4]. Thus (4.2) yields a *τ-independent* Poisson algebra: $\{L_{e_n}, L_{e_m}\}_\tau = \sum_j C^j_{nm} L_{e_{n+m+j}}(\tau)$.

The whole setup for (ξ^+, T_+) can be repeated for (ξ^-, T_-) by simply using the anti-holomorphic bases $\{\overline{e}_n\}$ and $\{\overline{\Omega}^m\}$. The resulting components of $T_- = \sum_{-\infty}^{\infty} L_{\overline{e}_n} \overline{\Omega}^n$ obey

$$\{L_{\overline{e}_n}, L_{\overline{e}_m}\}_\tau = L_{[\overline{e}_n, \overline{e}_m]} = \sum_j \overline{C}^j_{nm} L_{\overline{e}_{n+m+j}}, \quad \{L_{e_n}, L_{\overline{e}_m}\}_\tau = 0,$$

where the C^j_{nm} are the structure constants of $[e_n, e_m]$. There are two major advantages in dealing with L_{e_n} and $L_{\overline{e}_m}$ instead of $Q_{\xi^\pm}$ (especially on the quantum level):

(i) The algebraic structure of $\{L_{e_n}\}$ is related (by complex conjugation) to the structure of $\{L_{\overline{e}_m}\}$. Such a property does not hold for $\xi^\pm$: there is no canonical map which assigns ξ^+-fields to ξ^--fields because the conditions $\xi^\pm(\gamma^\pm_c) = 0$ are very different.

(ii) The algebra $\mathcal{L}(M)$ is generalized graded and different representations for central extensions of $\mathcal{L}(M)$ are available [4,10]. An analogous grading for C_M is not obvious.

Finally, we point out that our description differs from the "Euclidean version" of string theory [11]. There, the action S_{eucl} has a conformal symmetry with respect to a Riemannian metric. The corresponding conserved quantities $\tilde{L}_{e_n}$ represent $\mathcal{L}(M)$ on phase space, but there is no Lorentz structure present. In our description both $\mathcal{L}(M)$ and C_M are represented on the same phase space.

Acknowledgement

I would like to thank Ernst Binz for helpful discussions. Special thanks are due to Günter Schwarz for encouraging me to set up a rigorous Hamiltonian framework. This work was supported in part by the GradFöG Fund of Baden-Württemberg.

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